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Pregroups with Length Functions

Faisal H. Nesayef

Departpent of Mathematics, Faculty of Science, University of Kirkuk, Kirkuk, Iraq

Abstract: **Stallings [6] in 1971 introduced the concept of a pregroup. Subsequent work has been dose by Hoare [2], Nesayef [5], Chiswell [1], and many others. Five axioms were originally introduced by Stallings [6], namely P1 , P2 , P3 , P4 , and P5. It has been proved in [5] that P3 is a consequence of the other axioms.**

Keywords: **Archimedean elements, Defined product of elements, Length Functions, Pregroup, Universal Group.**

I. INTRODUCTION

In section one, we introduced the main concept and definition which we needed in later sections. In section two, we proved that some axioms are equivalent to the other ones.

We will also investigate some basic properties of Pregroups. Finally we show that the universal group of a Pregroup has a length function given by Lyndon [3]

II. LENGTH FUNCTION

Definition 2.1: A length function 1 1 an a group G , is a function given each element x of G a real number $|x|$, such that for all $x , y , z \in G$, the following axioms are satisfied

$A1'$ $|e| = 0$, e is the identity elements of G .

$A2$ $|x^{-1}| = |x|$

$A4$ $d(x,y) < d(y,z) \Longrightarrow d(x,y) = d(x,z)$, where $d(x,y) = \frac{1}{2}$ ($|x| + |y| - |xy^{-1}|$

Lyndon showed that $A4$ is equivalent to $d(x,y) \geq \min\ \{ d(y,z), d(x,z)\}$ and to

$d(y,z), d(x,z) \geq m \Longrightarrow d(x,z) \geq m$.

$A1', A2$ and $A4$ imply $|x| \geq d(x,y) = d(y,x) \geq 0$

Assuming, $A2$ and $A4$ only, it is easy to show that:

i. $e\ d(x,y) \geq |e|$

ii. $|x| \geq |e|$

iii. $d(x,y) \leq |x| - \frac{1}{2}\ |e|$, see [2]

$A3$ states that $d(x,y) \geq 0$ is deductible from $A1', A2$ and $A1'$ is a weaker version of the axiom:

$A1$ $|x| = 0$ if and only if $x = 1$ in G .

The following results are introduced by Lyndon [3].

(1) $d(xy,y) + d(x,y^{-1}) = |y|$

(2) $d(x,y^{-1}) + d(y,z^{-1}) \leq |y|$ Implies $|x\ y\ z| \leq |x| - |y| + |z|$

(3) $d(x, y^{-1}) + d(y, z^{-1}) \leq |y|$ Implies $d(xy, z^{-1}) = d(y, z^{-1})$

(4) $d(x, y) + d(x^{-1}, y^{-1}) \geq |x| = |y|$ Implies $|(xy^{-1})^2| \leq |xy^{-1}|)$

It follows from (2) that for any $x, y \in G$, $d(x, y) = |y| - d(x y^{-1}, y^{-1}) \leq |y|$ by A3.

Since $d(x, y) = d(y, x)$ we get $d(x, y) \leq \min\{|x|, |y|\}$

As state that $d(x, y) + d(x^{-1}, y^{-1}) > |x| = y \Rightarrow x = y$

Definition 2.2: A non-trivial element g of a group G is **called non-Archimedean** if $|g^2| \leq |g|$

Definition 2.3: Let G be a group with length function an element $x \neq 1$ in g is called Archimedean if $|x| \leq |x^2|$.

The following Axioms and results have added by Lyndon and others

$A0 \quad x \neq 1 \implies |x| < |x^2|$

$C0 \quad d(x, y)$ is always an integer

$C1 \quad x \neq 1, |x^2| \leq |x|$ *implies* $|x|$ is odd

$C2 \quad For\ no\ x\ is\ |x^2| = |x| + 1$

$C3 \quad if\ |x|\ is\ odd\ then\ |x^2| \geq |x|$

$C1' \quad if\ |x|\ is\ even\ and\ |x| \neq 0, then\ |x^2| > |x|$

$N0 \quad |x^2| \leq |x|\ implies\ x^2 = 1\ \ is\ x = x^{-1}$

$$N1^*\ G\ is\ general\ by\ \{x \in G : |x| \leq 1\}$$

3. Pregroups:

Stallings [6] introduced the fallowing construction of a Pregroup.

Definition 3.1: A Pregroup is a set P containing an element called the identity element of P, denoted by 1, a subset D of $P \times P$ and a mapping $D \to P$, when $(x, y) \to x y$ together with a map $i : P \to P$ when $i(x) = x^{-1}$, satisfying the following axioms. (We say that x y is defined if $(x, y) \in D$, i.e. $x y \in P$).

P1. For all $x \in P$, 1x and x1 are defined and 1x = x1 = x .

P2 . For all $x \in P$, $x^{-1} x = x^{-1} x = 1$

P3 . For all $x, y \in P$ if x y is defined , then $y^{-1} x^{-1}$ is defined and $(x y)^{-1} = y^{-1} x^{-1}$.

P4. Suppose that $x, y, z \in P$. If x y and y z are defined, then x (y z) is defined, is which case x (y z) = (x y) z.

P5. If w, x, y, z $\in$ P, and if w x, x y, y z, are all defined them either w (x y) or (x y) z is defined.

Proposition 3.2: Let P be a pregroup and $a, x \in P$. If a x is defined, them $a^{-1}(a x)$ is defined and $a^{-1}(a x) = x$.

Proof: By P2, we have $a^{-1} a$ is defined and equals 1. Thus by P4 and P1, we have $a^{-1}(a x)$ is defined and $a^{-1}(a x) = (a^{-1} a) x = x$.

The following propositions prove that P3 is a consequence of the other axioms.

Proposition 3.3: Let P be a pregroup and x, y, $\in$ P . If x y is defined them $y^{-1} x^{-1}$ is defined and $(x y)^{-1} = y^{-1} x^{-1}$

Proof: Suppose x y is defined . Then $x y \in P$ and $(x y)^{-1} \in P$.

Consider: $x^{-1}, x y, (x y)^{-1}$:

$x^{-1}(x y)$ and $(x y)(x y)^{-1}$ are defined .

Since $x^{-1}[(x\,y)(x\,y)^{-1}]$ is defined and equals to x^{-1} then by P4, we have :

$[x^{-1}(x\,y)](x\,y)^{-1}$ is also defined and $= x^{-1}[(x\,y)(x\,y)^{-1}] = x^{-1}$

By P4 again : $y(x\,y)^{-1} = x^{-1}$

Now consider : $y^{-1}, y, (x\,y)^{-1}$: $y^{-1}\,y$ and $y(x\,y)$ are both defined .

Since $[y^{-1}\,y](x\,y)^{-1}$ is defined and $= (x\,y)^{-1}$.

Then by P4: $y^{-1}[y(x\,y)^{-1}]$ is also defined and $= (x\,y)^{-1}$.

Definition 3.4: Let P be Pregroup. A **word** in P is an n-tuple: $(x_1 \ldots x_4)$ of elements of P , for some $n \geq 1$, n is called the **length** of the word.

Definition 3.5 : A word $(x_1 \ldots x_n)$ is said to be reduced if $x_i\,x_{i+1}$ is not defined for any $1 \leq i \leq n-1$.

Let $P_0 = \{x \in P : x\,y$ and $y\,x$ are defined for all $y \in P\}$. We call P_o the **core** of P.

Proposition 3.6: P_o is a subgroup.

Proof: Suppose $x \in P_o$. By the definition of P_o : $x\,y$, $y\,x$ are defined for all $y \in P$ and so $y^{-1}x^{-1}$ and $y^{-1}x$ are both defined , so $x^{-1} \in P$.

Suppose $x\,y \in P_o$. $x\,y$, $y\,z$ and $x(y\,z)$ are all defined for all $z \in P$.

By P4: $(x\,y)\,z$ defined for all $z \in P_o$.

Definition 3.7: Let P be any Pregroup. The **Universal group** U (P) is the set of all equivalence classes of reduced words.

Theorem 3.6 $|\;| : U(P) \to P$ satisfies the following axiom:

$A1'$ $|1| = 0$

$A2$ $|g| = |g^{-1}|$, $g \,\varepsilon\, U(P)$

$A4'$ $d(g,h), d(h,k) > s \geq 0 \to d(g,k) \geq s$, where s is half an integer and

$2d(g,h) = |g| + |h| - |gh^{-1}|$, for all , $h,k \,\varepsilon\, U(P)$.

Proof $A1'$, $A2$ are obvious, and clearly $d(g,h) \geq 0$ so we shall prove $A4'$

Let $g,h,k \,\varepsilon\, U(P)$ the result is trivial if any one of $|g|$, $|h|, |k|$ is zero, because

$d(g,h) \geq 0$ so let $g = x_1 \ldots x_n$, $|g| = n \geq 1$

$h = y_1 \ldots y_m$, $|h| = m \geq 1$, and $k = z_1 \ldots z_\ell$, $|k| = \ell \geq 1$, be reduced , where x_1 , y_1 , and $z_1 \notin A_0$

Suppose $d(g,h), d(h,k) > s$

Case 1 s is an integer

$gh^{-1} = x_1 \ldots (x_{n-s}\, a_s\, y_{m-s}^{-1}) \ldots y_1^{-1}$, such that $a_{s+1} = x_{n-s}\, a_s\, y_{m-s}^{-1}$ is defined , where

$a_s = x_{n-s+1} \ldots x_n\, y_m^{-1} \ldots y_{m-s+1}^{-1}$ and $a_0 = 1$.

Similarly $hk^{-1} = y_1 \ldots y_{m-s}\, b_s\, z_{\ell-s}^{-1} \ldots z_1^{-1}$, such that $b_{s+1} = y_{m-s}\, b_s\, z_{\ell-s}^{-1}$ is defined , where

$b_s = y_{m-s+1} \ldots y_m\, z_\ell^{-1} \ldots z_{\ell-s+1}^{-1}$ and $b_0 = 1$

$$gk^{-1} = x_1 \ldots x_n\, y_m^{-1} \ldots y_{m-s}^{-1}\, y_{m-s} \ldots y_m\, z_\ell^{-1} \ldots . z_1^{-1}$$

$$= x_1 \ldots \underbrace{(x_{n-s} \ldots x_n\, y_m^{-1} \ldots y_{m-s}^{-1})}_{a_{s+1}}\, \underbrace{(y_{m-s} \ldots y_m\, z_\ell^{-1} \ldots . z_{\ell-s}^{-1})}_{b_{s+1}} \ldots z_1^{-1}$$

Then $|gk^{-1}| \leq n-s-1+1+1+\ell-s-1 \leq n+\ell-2s$ i.e $d(g,k) \geq s$

case 2 s is not an integer

Suppose $(g,k), d\,(h,k) > s = r-\frac{1}{2}, r \geq 1$

$gk^{-1} = x_1 \dots x_{n-r}\, a_r\, y_{m-r}^{-1} \dots y_1^{-1}$, where a_r is defined and equals

$x_{n-r+1} \dots x_n\, y_m^{-1} \dots y_{m-r+1}^{-1}$, and either $x_{n-r}\, a_r$ is defined (1)

Or $a_r\, y_{m-r}^{-1}$ is defined (2)

Similarly $hk^{-1} = y_1 \dots y_{n-r}\, b_r\, z_{\ell-r}^{-1} \dots z_1^{-1}$, b_r is defined and

$b_r = y_{m-r+1} \dots y_m\, z_\ell^{-1} \dots z_{\ell-r+1}^{-1}$, and either $y_{m-r}\, b_r$ is defined (3)

Or $b_r\, z_{\ell-r}^{-1}$ is defined (4)

Suppose (2) and (3) hold:

If $a_r = (n_{n-r+1}\, a_{r-1}) y_{m-r+1}^{-1}$, then apply P5 on :

$(x_{n-r+1}\, a_{r-1})^{-1}, (x_{n-r+1}\, a_{r-1}) y_{m-r+1}^{-1}, y_{m-r}^{-1}\, y_{m-r}\, b_r$

Since the product of the first three terms is not defined, then

$(x_{n-r+1}\, a_{r-1}) y_{m-r+1}^{-1}, y_{m-r}^{-1}\, y_{m-r}\, b_r$ is defined , i.e. $a_r\, b_r$ is defined .

If $a_r\, x_{n-r+1}\, (a_{r-1}\;\; y_{m-r+1}^{-1})$, then apply P5 on $y_{n-r+1}^{-1}, y_{n-r+1}\, (a_{r-1}\;\; y_{m-r+1}^{-1}), y_{m-r}^{-1}\, y_{m-r}\, b_r$

Since $a_{r-1}\;\; y_{m-r+1}^{-1}\, y_{m-r}^{-1}$ is not defined, then $x_{n-r+1}(a_{r-1}\;\; y_{m-r+1}^{-1}) y_{m-r}^{-1}\, y_{m-r}\, b_r$ is defined, i.e $a_r\;\; b_r$ is defined.

Put $a_r\;\; b_r\;\; = c_r$,

$gk^{-1} = x_1 \dots x_n\, y_m^{-1} \dots y_{m-r+1}^{-1}\, y_{m-r+1} \dots y_m\, z_\ell^{-1} \dots . z_1^{-1}$

$= x_1 \dots x_{n-r} \dots a_r\, b_r z_{\ell-r}^{-1} \dots z_1^{-1} = x_1 \dots x_{n-r} \dots c_r\, z_{\ell-r}^{-1} \dots z_1^{-1}$

i.e. $|gk^{-1}| \leq n-r+1+\ell-r = n+\ell-2r+1$

Then $d\,(g,k) \geq r-\frac{1}{2} = s$

If (1) holds , then $gk^{-1} = x_r \dots (x_{n-r}\, a_r)\, b_r\, z_{\ell-r}^{-1} \dots z_1^{-1}$, i.e.

$|gk^{-1}| \leq n-r+1+\ell-r$. thus $d\,(g,k) \geq r-\frac{1}{2} = s$

If (4) holds, then $gk^{-1} = x_1 \dots x_{n-r}\, a_r\, (b_r\, z_{\ell-r}^{-1})\, z_1^{-1}$, i.e.

$|gk^{-1}| \leq n-r+1+\ell-r$, so again $d\,(g,k) \geq r-\frac{1}{2} = s$

Therefore $A4'$ is satisfied

III. CONCLUSION

This paper shows that the Universal group of a pregroup can be occupied with a length function defined by Lyndon [3]. Therefore it will have all the combinatorial group properties, which are open for investigations.

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REFERENCES

[1] Chiswell, I. M ; Length Function and Pregroups, Proceedings of Edinburgh Mathematical Society (1987), 30, 57 – 67.

[2] Hoare, A. H. M.; Nielson methods in groups with length functions, Math. Scand, (1981), 153 – 164.

[3] Lyndon, R. C; Length Functions in Groups, Math. Scand, 12, (1963), 209 – 234.

[4] Nesayef, F.H.; Decomposition of Pregroups, International Journal of Scientific & Engineering Research, Volume 5, Issue 1, January-2014, (1035-1037).

[5] Nesayef, F. H. ; Groups generated by element of length zero and one, Ph. D. Thesis, University of Birmingham, UK, 1983.

[6] Stallings, J. R.; Group theory and three dimensional manifolds, New Haven. (12), University Press, 1971.